J. Fluid Mech. (1972), *vol.* 54, *part* 3, *pp.* 507–520
Printed in Great Britain

Nonlinear energy transfer in gravity–capillary wave spectra, with applications

By G. R. VALENZUELA AND M. B. LAING
Naval Research Laboratory, Washington, D.C.

(Received 7 February 1972)

The energy flux in gravity–capillary wave spectra has been obtained using Hasselmann's (1962) perturbation analysis for a homogeneous Gaussian sea. As expected, resonant interactions now appear at second order, and a third-order perturbation analysis shows that energy is redistributed from waves with intermediate wavelengths (in the neighbourhood of 1·7 cm) toward gravity and capillary waves. Numerical computations are also obtained for the energy flux and the interaction time of a sharply peaked spectrum consisting of wavenumbers concentrated around a single wavenumber, superposed on a smooth background spectrum. The range of validity of the inviscid results is discussed.

1. Introduction

The resonant nonlinear interactions of surface gravity waves are weak, of third order, and their dynamics for discrete interactions have been investigated by Phillips (1960), Longuet-Higgins (1962) and Benney (1962). The energy transfer in gravity wave spectra has been found to be of fourth order by Hasselmann (1962, 1963*a*, *b*), using a fifth-order perturbation analysis, and the energy of the waves is redistributed from the waves with intermediate wavenumbers to gravity waves of lower and higher wavenumbers.

For gravity–capillary waves the dispersion relation $\omega = \omega(k)$ connecting the radian frequency ω and the wavenumber k becomes concave for wavenumbers for which surface tension becomes important and this allows resonant interactions among gravity–capillary waves at second order. Energy is exchanged among a triad of waves; for discrete interactions the dynamics have been investigated in detail by McGoldrick (1965, 1970) and Simmons (1969). The general form of the transfer expression for triad interactions in case of a continuum of waves is well known from solid-state and plasma physics and is given by Hasselmann (1966, 1968).

Wave tank measurements of slope spectra by Cox (1958) and by Wright & Keller (1971), and open-seas height spectral measurements by Valenzuela, Laing & Daley (1971) show a 'dip' in the spectrum (a marked reduction of amplitude) for wavelengths in the neighbourhood of 1·7 cm for wind speeds smaller than 6–7 m/s. To discover whether this 'dip' in the spectrum for 1·7 cm waves is the result of the dynamics of the nonlinear wave–wave resonant interactions we have applied Hasselmann's (1962) inviscid formulation to obtain the energy flux in gravity–capillary wave spectra; now, of course, we must include surface tension.

In this formulation we preserve the assumption that the linear approximation for the sea surface is homogeneous, stationary and Gaussian. Now the resonant interactions are stronger, of second order, and they should be most important in the overall energy balance of the wind–sea interaction. The effect of viscosity, as will be discussed later, will restrict the validity of the inviscid results to wavelengths in the neighbourhood of 1·7 cm.

If we consider the form of the energy-transfer expression for gravity–capillary waves of small slopes, further insight should be gained into the growth of the spectrum in its initial stages. For example, McGoldrick (1965) found that for discrete wave interactions typical interaction times for 1–2 cm wavelengths are smaller than the wind growth rates. Thus, wave–wave interactions should play an important role in the development of the spectrum after its initial generation by Phillips' (1957) resonance mechanism.

2. Perturbation analysis

The analysis used in this formulation is similar to that given by Hasselmann (1962) and the notation used will be that of his paper. The analysis applies to irrotational motion of a horizontally unbounded fluid of infinite depth ($z = -\infty$) with a free surface at $z = \zeta(x,y;t)$, where x, y and z denote Cartesian co-ordinates, with x, y in the horizontal plane, and t is time. The velocity potential $\phi(x,y,z;t)$ and the surface deviation ζ satisfy the following well-known differential equations which now include surface tension:

$$\nabla^2\phi = 0 \quad \text{for} \quad z < \zeta, \tag{2.1}$$

$$\frac{\partial\zeta}{\partial t} - \frac{\partial\phi}{\partial z} + \tilde{\nabla}\zeta.\tilde{\nabla}\phi = 0 \quad \text{at} \quad z = \zeta, \tag{2.2}$$

$$\frac{\partial\phi}{\partial t} + g\zeta + \tfrac{1}{2}(\nabla\phi)^2 - \frac{T'}{\rho}\left\{\frac{\nabla.\nabla\zeta}{N} - \frac{\nabla\zeta.\nabla\nabla\zeta.\nabla\zeta}{N^3}\right\} = 0 \quad \text{at} \quad z = \zeta, \tag{2.3}$$

where $\tilde{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, $N = \left[1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right]^{\frac{1}{2}}$,

ρ is the water density, T' is the surface tension and g is gravity; ϕ and ζ satisfy given initial conditions for $t = 0$.

Here, we shall not describe the perturbation analysis in detail and we merely point out some of the important steps and some of the differences between this analysis and that given by Hasselmann (1962). The boundary conditions (2.2) and (2.3) are, as usual, expanded in a Taylor series about $z = 0$, and the velocity potential ϕ and the surface deviation ζ are expanded in a series:

$$\phi = {}_1\phi + {}_2\phi + {}_3\phi + \dots \tag{2.4}$$

and

$$\zeta = {}_1\zeta + {}_2\zeta + {}_3\zeta + \dots. \tag{2.5}$$

Since the wave field is taken to be a homogeneous and Gaussian random function, at every order ϕ and ζ can be approximated by Fourier sums:

$$ {}_n\phi = \sum_{\mathbf{k}} {}_n\phi_{\mathbf{k}}(t)\, e^{kz} e^{i\mathbf{k}.\mathbf{x}} \tag{2.6}$$

and

$$ {}_n\zeta = \sum_{\mathbf{k}} {}_nZ_{\mathbf{k}}(t)\, e^{i\mathbf{k}.\mathbf{x}}, \tag{2.7}$$

where $k = |\mathbf{k}|$ and $\mathbf{x} = (x, y)$. The complex random Fourier coefficients in (2.6) and (2.7) are independent for each wavenumber and since ϕ and ζ are real quantities it follows that ${}_n\phi_{\mathbf{k}} = ({}_n\phi_{-\mathbf{k}})^*$ and ${}_nZ_{\mathbf{k}} = ({}_nZ_{-\mathbf{k}})^*$, where the asterisk denotes the complex conjugate.

Substituting (2.4)–(2.7) into the boundary condition expansions about $z = 0$ and collecting terms of the same order, to first order we obtain the familiar differential equations

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{k}}^2\right) {}_1\phi_{\mathbf{k}}(t) = 0 \tag{2.8}$$

and

$$(g + Tk^2)\, {}_1Z_{\mathbf{k}}(t) + \partial\, {}_1\phi_{\mathbf{k}}(t)/\partial t = 0. \tag{2.9}$$

The solutions of (2.8) and (2.9) are well known and are

$$ {}_1\phi_{\mathbf{k}}(t) = {}_1\Phi_{\mathbf{k}}^{+} e^{-i\omega_{\mathbf{k}}t} + {}_1\Phi_{\mathbf{k}}^{-} e^{i\omega_{\mathbf{k}}t}, \tag{2.10}$$

$$ {}_1Z_{\mathbf{k}}(t) = {}_1Z_{\mathbf{k}}^{+} e^{-i\omega_{\mathbf{k}}t} + {}_1Z_{\mathbf{k}}^{-} e^{i\omega_{\mathbf{k}}t}, \tag{2.11}$$

with ${}_1Z_{\mathbf{k}}^{\pm} = [\pm i\omega_{\mathbf{k}}/(g+Tk^2)]\, {}_1\Phi_{\mathbf{k}}^{\pm}$ and $\omega_{\mathbf{k}}^2 = gk + Tk^3$, the dispersion relation for gravity–capillary waves, and $T = T'/\rho$. The signs in ${}_1\Phi_{\mathbf{k}}^{\pm}$ and ${}_1Z_{\mathbf{k}}^{\pm}$ indicate the direction of propagation of a wave of wavenumber $\mathbf{k}$.

The second-order velocity potential satisfies the inhomogeneous harmonic differential equation

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{k}}^2\right) {}_2\phi_{\mathbf{k}} = \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}\\ s_1, s_2}} D_{\mathbf{k}_1,\mathbf{k}_2}^{s_1,s_2}\, {}_1\Phi_{\mathbf{k}_1}^{s_1}\, {}_1\Phi_{\mathbf{k}_2}^{s_2}\, e^{-i(s_1\omega_{\mathbf{k}_1}+s_2\omega_{\mathbf{k}_2})t}, \tag{2.12}$$

where $s_1, s_2 = \pm$, the summation is over $\mathbf{k}$, s_1 and s_2, and

$$D_{\mathbf{k}_1,\mathbf{k}_2}^{s_1,s_2} = \frac{i}{2}\left\{(\omega_1+\omega_2)(k_1k_2 - \mathbf{k}_1.\mathbf{k}_2) + \omega_1\omega_2(\omega_1+\omega_2)\left(\frac{k_1}{g+Tk_1^2} + \frac{k_2}{g+Tk_2^2}\right)\right.$$
$$\left. - g(+Tk^2)\left[\frac{\omega_1(k_1^2+\mathbf{k}_1.\mathbf{k}_2)}{g+Tk_1^2} + \frac{\omega_2(k_2^2+\mathbf{k}_1.\mathbf{k}_2)}{g+Tk_2^2}\right]\right\},$$

in which we have used the abbreviation $\omega_1 = s_1\omega_{\mathbf{k}_1}$ and $\omega_2 = s_2\omega_{\mathbf{k}_2}$.

The dispersion relation for gravity–capillary waves, $\omega_{\mathbf{k}}^2 = gk + Tk^3$, is concave toward the capillary region, allowing the possibility of resonant excitations of (2.12), that is $\omega_{\mathbf{k}}^2 = (s_1\omega_{\mathbf{k}_1} + s_2\omega_{\mathbf{k}_2})^2$ and $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ can be satisfied simultaneously; see McGoldrick (1965). Thus non-steady harmonic solutions (with amplitude increasing with time) appear at second order for gravity–capillary waves, whereas for gravity waves they appear at third order, see Hasselmann (1962).

At third order, the velocity potential for gravity–capillary waves satisfies a more complicated inhomogeneous harmonic differential equation. The excitation now is partly composed of already non-steady harmonic solutions:

$$\left(\frac{\partial^2}{\partial t^2}+\omega_{\mathbf{k}}^2\right){}_3\phi_{\mathbf{k}} = \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}\\ s_1,\, s_1,\, s_3}} D^{s_1, s_2, s_3}_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3}\, {}_1\Phi^{s_1}_{\mathbf{k}_1}\, {}_1\Phi^{s_2}_{\mathbf{k}_2}\, {}_1\Phi^{s_3}_{\mathbf{k}_3}$$

$$\times \mathscr{I}_1(\omega_{\mathbf{k}_2+\mathbf{k}_3}, -s_2\omega_{\mathbf{k}_2}-s_3\omega_{\mathbf{k}_3}; t)\, e^{-is_1\omega_{\mathbf{k}_1}t} + \text{non-contributing terms}, \tag{2.13}$$

where $D^{s_1, s_2, s_3}_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3} = 2D^{\omega_1, \omega_2+\omega_3}_{\mathbf{k}_1, \mathbf{k}_2+\mathbf{k}_3} D^{s_2, s_3}_{\mathbf{k}_2, \mathbf{k}_3}$ and $\mathscr{I}_1(\omega, \omega'; t)$ is the solution of

$$d^2\psi/dt^2+\omega^2\psi = e^{i\omega' t} \quad \text{with} \quad \psi = d\psi/dt = 0 \quad \text{at} \quad t = 0.$$

3. The energy transfer

For gravity–capillary waves the average energy per unit area of the sea is given by

$$E = \tfrac{1}{2}\rho\overline{\int_{-\infty}^{\zeta} (\nabla\phi)^2\, dz}+\tfrac{1}{2}\rho g\overline{\zeta^2}+T'\overline{[(1+\tilde{\nabla}\zeta.\tilde{\nabla}\zeta)^{\frac{1}{2}}-1]}, \tag{3.1}$$

where the overbar indicates an ensemble average.

Expanding (3.1) in a Taylor series about $z = 0$ for a homogeneous sea, substituting the expansions (2.4) and (2.5) for ϕ and ζ and collecting terms of equal order we find that the mean energy per unit area of the sea is given by the perturbation series

$$E = {}_2E+{}_3E+{}_4E+{}_5E+{}_6E+\dots, \tag{3.2}$$

in which the odd terms vanish for a Gaussian sea and the first two even terms are given by

$${}_2E = \tfrac{1}{2}\rho\left(\overline{{}_1\phi\frac{\partial_1\phi}{\partial z}}\right)+\tfrac{1}{2}\rho g\overline{{}_1\zeta^2}+\tfrac{1}{2}T'\left(\overline{\tilde{\nabla}_1\zeta.\tilde{\nabla}_1\zeta}\right) = \int_{-\infty}^{\infty}\int F(\mathbf{k})\, d\mathbf{k} \tag{3.3}$$

and

$${}_4E = \tfrac{1}{2}\rho\left(\overline{{}_2\phi\frac{\partial_2\phi}{\partial z}}\right)+\tfrac{1}{2}\rho\left(\overline{{}_1\phi\frac{\partial_3\phi}{\partial z}}\right)+\tfrac{1}{2}\rho\left(\overline{{}_3\phi\frac{\partial_1\phi}{\partial z}}\right)$$

$$+\tfrac{1}{2}\rho g\overline{{}_2\zeta^2}+\rho g\overline{{}_1\zeta_3\zeta}+\tfrac{1}{2}T'\left(\overline{\tilde{\nabla}_2\zeta.\tilde{\nabla}_2\zeta}\right)+T'\left(\overline{\tilde{\nabla}_1\zeta.\tilde{\nabla}_3\zeta}\right), \tag{3.4}$$

where $d\mathbf{k} = dk_x dk_y$ and $F(\mathbf{k})$ is the two-dimensional energy spectrum for wave components travelling in the positive $\mathbf{k}$ direction and for a homogeneous, stationary Gaussian process completely determines the surface in the linear approximation.

The problem then reduces to finding the time-dependent covariance products which for gravity–capillary waves appear to order ${}_4E$; for gravity waves they did not appear until the higher order ${}_6E$. The time-dependent covariance products contributing to ${}_4E$ can be found using (2.12) and (2.13) with the asymptotic formulae given in Hasselmann (1962). Since the procedure followed is quite similar to that given by Hasselmann, here we shall only point out that the time-dependent terms contributing to ${}_4E$ increase linearly with time and are expressible in terms of $F(\mathbf{k})$, the two-dimensional energy spectrum for the linear approximation to the sea.

After performing the lengthy algebra, keeping in mind that we only include time-dependent terms of order up to ${}_4E$, we obtain the following expression for the energy transfer in gravity–capillary wave spectra, which has been made to satisfy the laws of conservation of energy and momentum:

$$\begin{aligned}\frac{\partial F(\mathbf{k}_3)}{\partial t} = \frac{\pi k_3}{2\rho\omega_{\mathbf{k}_3}^4}\int_{-\infty}^{\infty}\int \frac{|D^{+,+}_{\mathbf{k}_1,\mathbf{k}_2}|^2}{k_1 k_2}\,\omega_{\mathbf{k}_3}\{\omega_{\mathbf{k}_3}F(\mathbf{k}_1)F(\mathbf{k}_2)-\omega_{\mathbf{k}_2}F(\mathbf{k}_3)F(\mathbf{k}_1)-\omega_{\mathbf{k}_1}F(\mathbf{k}_3)F(\mathbf{k}_2)\}\\ \times\,\delta(\omega_{\mathbf{k}_3}-\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}_1})\,dk_{x_1}\,dk_{y_1}+\frac{\pi k_3}{\rho\omega_{\mathbf{k}_3}^4}\int_{-\infty}^{\infty}\int\frac{|D^{+,-}_{\mathbf{k}_1,-\mathbf{k}_2}|^2}{k_1 k_2}\,\omega_{\mathbf{k}_3}\\ \times\{\omega_{\mathbf{k}_3}F(\mathbf{k}_1)F(\mathbf{k}_2)-\omega_{\mathbf{k}_2}F(\mathbf{k}_3)F(\mathbf{k}_1)+\omega_{\mathbf{k}_1}F(\mathbf{k}_3)F(\mathbf{k}_2)\}\\ \times\,\delta(\omega_{\mathbf{k}_3}-\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_1})\,dk_{x_1}dk_{y_1},\end{aligned} \tag{3.5}$$

where δ is the Dirac delta function. The first integral in (3.5) is the contribution by sum resonant interaction, $\mathbf{k}_1+\mathbf{k}_2 = \mathbf{k}_3$, and the second integral is the contribution from difference resonant interactions, $\mathbf{k}_2-\mathbf{k}_1 = \mathbf{k}_3$ and $\mathbf{k}_1-\mathbf{k}_2 = \mathbf{k}_3$.

4. Interpretation and results

As expected, the transfer of energy in gravity–capillary wave spectra is among a triad of waves, energy being transferred from two active components to a third passive component. Two types of interactions participate, sum resonant interactions, which satisfy $\mathbf{k}_1+\mathbf{k}_2 = \mathbf{k}_3$ and $\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2} = \omega_{\mathbf{k}_3}$ simultaneously, and difference resonant interactions, which satisfy $\mathbf{k}_2-\mathbf{k}_1 = \mathbf{k}_3$ and $\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2} = \omega_{\mathbf{k}_3}$ simultaneously.

A more quantitative investigation of the energy transfer is obtained by evaluating (3.5) for a polar two-dimensional spectrum $S(k,\alpha)$, which is related to the energy spectrum by

$$S(k,\alpha) = kF(\mathbf{k})/2\rho\omega_{\mathbf{k}}^2. \tag{4.1}$$

We put $S(k,\alpha) = S(k)\,S(\alpha)$ with $S(k) = 0{\cdot}01\,k^{-4}\exp\{-Ck_m/k\}$, where $k_m = (g/T)^{\frac{1}{2}}$ and C is a constant. Various spreading factors $S(\alpha)$ are used and have been normalized in the half plane.

On account of the δ functions, the energy transfer is a line integral in which the path of integration is found from the resonant interaction condition

$$\omega_{\mathbf{k}_2}\pm\omega_{\mathbf{k}_1} = \omega_{\mathbf{k}_3}, \quad \mathbf{k}_2\pm\mathbf{k}_1 = \mathbf{k}_3, \tag{4.2}$$

where the upper signs apply for sum resonant interactions and the lower signs apply for difference interactions.

Denoting the angle between $\mathbf{k}_1$ and $\mathbf{k}_3$ by β, the following cubic equation is obtained from (4.2):

$$\begin{aligned}\cos^3\beta\mp\cos^2\beta\,\frac{[2+3(\kappa_1^2+\kappa_3^2)]}{2\kappa_1\kappa_3}+\cos\beta\,\frac{[3(\kappa_1^2+\kappa_3^2)+4(\kappa_1^2+\kappa_3^2)+1]}{4\kappa_1^2\kappa_3^2}\\ \pm\frac{\{\kappa_3(1+\kappa_3^2)+\kappa_1(1+\kappa_1^2)\mp 2[\kappa_1\kappa_3(1+\kappa_1^2)(1+\kappa_3^2)]^{\frac{1}{2}}\}^2}{8\kappa_1^3\kappa_3^3}\mp\frac{(\kappa_1^2+\kappa_3^2)(1+\kappa_1^2+\kappa_3^2)^2}{8\kappa_1^3\kappa_3^3} = 0,\end{aligned} \tag{4.3}$$

where the upper signs again apply for sum resonant interactions and the lower signs apply for difference resonant interactions, $\kappa_1 = k_1/k_m$ and $\kappa_3 = k_3/k_m$.

The solution of (4.3) was given in graphical form for sum resonant interactions by McGoldrick (1965). The point to be made here is that for a fixed value of k_3 the sum resonant interactions contribute for $k_3 \geqslant 2^{\frac{1}{2}}k_m$ and the difference resonant interactions should be active for all k_3, except that for gravity waves the other two components should be in the capillary region and for k_3 in the capillary region the contribution will come for components in both the gravity region and in the capillary region. Thus, in general, viscosity should be important to the energy transfer for k_3 both in the gravity region and in the capillary region.

The actual expression used for the computation of the energy flux is given below:

$$\begin{aligned}\frac{\partial S(k_3,\alpha_3)}{\partial t} = &\int_0^\infty \sum_{j=1}^{2} T^{(+)}\left\{\frac{k_3}{k_2} S(k_1,\alpha_1^{(j)})\, S(k_2,\alpha_2^{(j)}) - \frac{\omega_{\mathbf{k}_3}}{\omega_{\mathbf{k}_2}} S(k_1,\alpha_1^{(j)})\, S(k_3,\alpha_3) - \frac{k_1\omega_{\mathbf{k}_3}}{k_2\omega_{\mathbf{k}_1}} S(k_2,\alpha_2^{(j)})\, S(k_3,\alpha_3)\right\} dk_1 \\ &+ 2\int_0^\infty \sum_{j=1}^{2} T^{(-)}\left\{\frac{k_3}{k_2} S(k_1,\alpha_1^{(j)})\, S(k_2,\alpha_2^{(j)}) - \frac{\omega_{\mathbf{k}_3}}{\omega_{\mathbf{k}_2}} S(k_1,\alpha_1^{(j)})\, S(k_3,\alpha_3) + \frac{k_1\omega_{\mathbf{k}_3}}{k_2\omega_{\mathbf{k}_1}} S(k_2,\alpha_2^{(j)})\, S(k_3,\alpha_3)\right\} dk_1, \end{aligned} \quad (4.4)$$

where

$$T^{(+)} = \begin{cases} \dfrac{2\pi\omega_{\mathbf{k}_1}^2\,\omega_{\mathbf{k}_2}^3\,|D_{\mathbf{k}_1,\mathbf{k}_2}^{+,+}|^2}{(g+3Tk_2^2)\,k_1^2\,\omega_{\mathbf{k}_2}^4\,|\sin\beta|} & \text{for} \quad |\cos\beta| < 1, \\ 0 & \text{for} \quad |\cos\beta| \geqslant 1, \end{cases}$$

and

$$T^{(-)} = \begin{cases} \dfrac{2\pi\omega_{\mathbf{k}_1}^2\,\omega_{\mathbf{k}_2}^3\,|D_{\mathbf{k}_1,-\mathbf{k}_2}^{+,-}|^2}{(g+3Tk_2^2)\,k_1^2\,\omega_{\mathbf{k}_3}^4\,|\sin\beta|} & \text{for} \quad |\cos\beta| < 1, \\ 0 & \text{for} \quad |\cos\beta| \geqslant 1. \end{cases}$$

As in the case of gravity waves, the transfer coefficients $T^{(+)}$ have integrable singularities for $\sin\beta = 0$. The sums in (4.4) over the index j account for the two possible angular solutions which can occur owing to the ambiguity in the sign of β in (4.3). For the sum resonant interactions the integrand has two integrable singularities for each value of $\mathbf{k}_3$ except $k_3 = 2^{\frac{1}{2}}k_m$, for which there is a single discontinuity at $k_1 = k_m/2^{\frac{1}{2}}$ and the integrand is zero for all other k_1. For the difference resonant interaction the integrand has a single integrable singularity for each value of k_3.

The main contribution to the energy transfer is from wave components almost parallel to each other and these components in general have different wavenumbers, except for difference resonance interactions and for $k_3 = k_m/2^{\frac{1}{2}}$, where $k_1 = k_m/2^{\frac{1}{2}}$ and $k_2 = 2^{\frac{1}{2}}k_m$, and for sum resonant interactions and for $k_3 = 2^{\frac{1}{2}}k_m$, where $k_1 = k_2 = k_m/2^{\frac{1}{2}}$.

Numerical results have been obtained from (4.4) with a CDC-3600 machine, using a variable-interval trapezoidal integration scheme to handle the integrable singularities at $\sin\beta = 0$. A minimum interval $10^{-6}\times\frac{1}{2}$ of k_m was used for k_1, and away from the singularities the interval was allowed to increase to $0{\cdot}1\,k_m$, always keeping the integration error to less than $0{\cdot}1\,\%$ at each step. The path of integration was determined from (4.3) for a fixed k_3; $\cos\beta$ was determined for each k_1 within 0 and $20\,k_m$; we have taken $g = 980\,\mathrm{cm/s^2}$ and $T = 74\,\mathrm{cm^3/s^2}$.

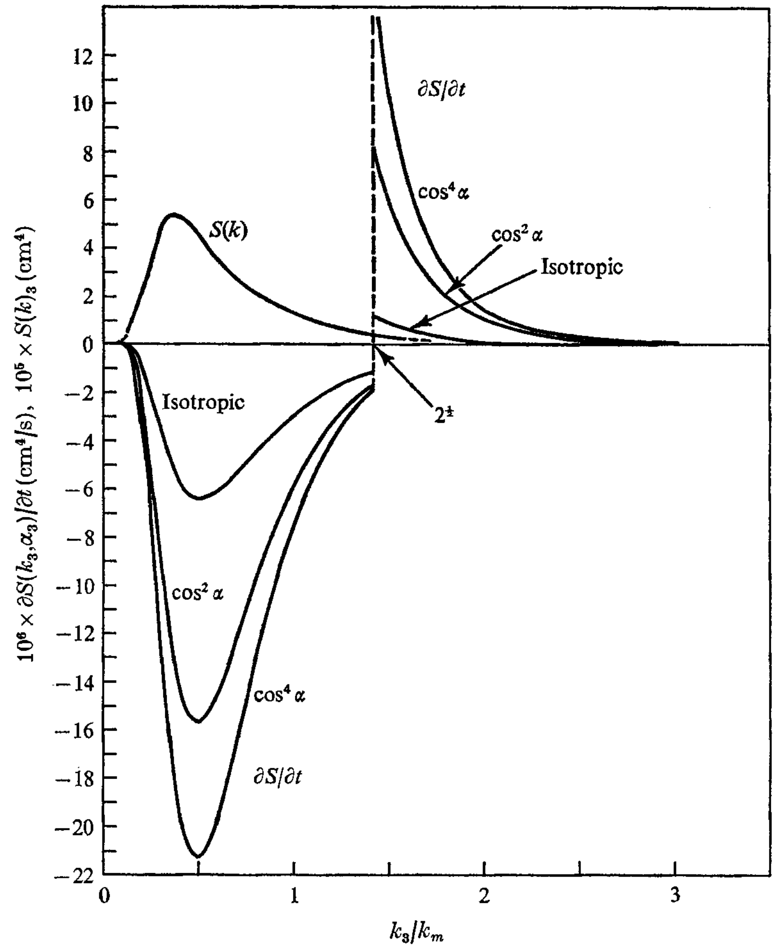

FIGURE 1. Energy transfer for a gravity–capillary wave spectrum with different spreading factors. For the $\cos^4\alpha$ spreading factor, the energy flux curve has been truncated near the discontinuity at $k_3 = 2^{\frac{1}{2}}k_m$. $\alpha_3 = 0$, $S(k) = 10^{-2}k^{-4}\exp\{-1{\cdot}5\,k_m/k\}$.

In figure 1 the energy flux is shown for various spreading factors for $C = 1{\cdot}5$ in the exponential factor of the radial part of the spectrum. As in the case of gravity waves, see Hasselmann (1963*b*), the energy transfer is greatest for the most directive spectrum. The energy flux now has a discontinuity at $k_3 = 2^{\frac{1}{2}}k_m$ which is connected with the instability of 2·44 cm wavelengths. Energy is transferred to the second harmonic with wavelengths of 1·22 cm; see Pierson & Fife (1961), McGoldrick (1965) and Simmons (1969).

The general shape of the energy-transfer curve for gravity–capillary waves is similar to that found for gravity waves by Hasselmann, except for the discontinuity at $k_3 = 2^{\frac{1}{2}}k_m$. Energy is principally redistributed from intermediate wavenumbers, in our case near k_m, toward smaller and larger wavenumbers. Thus the 'dip' in the spectrum for light winds could be a result of the energy transfer produced by wave–wave resonant interactions. In figure 2, the energy flux in various directions with respect to the sea spectrum is shown for a $\cos^2\alpha$

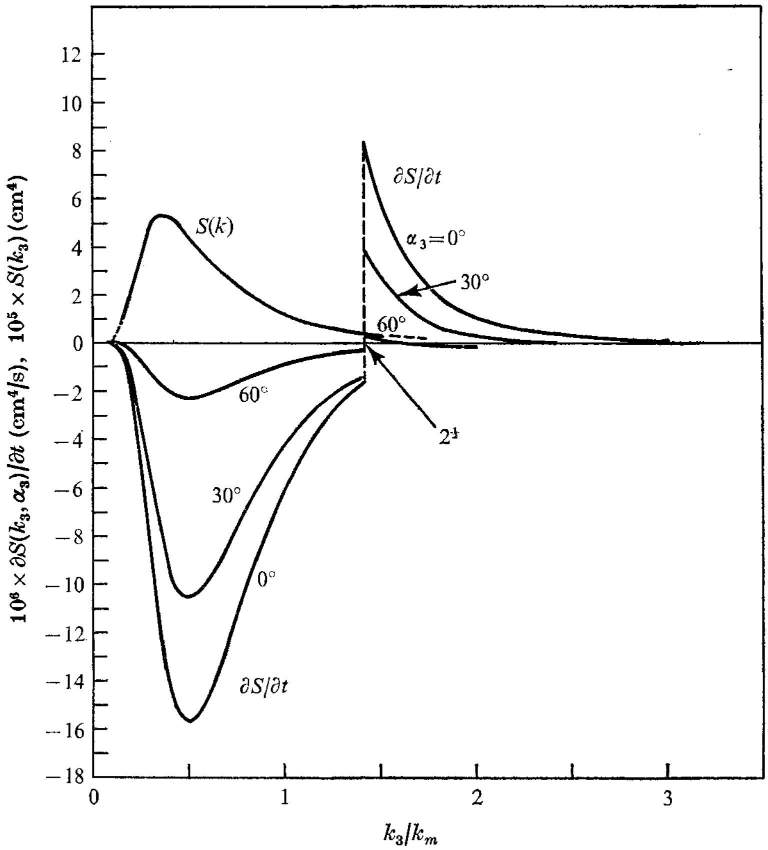

FIGURE 2. Energy transfer for a gravity–capillary wave spectrum with $\cos \alpha^4$ spreading factor for several directions. $S(\alpha) = 2/\pi \cos^2 \alpha$, $S(k) = 10^{-2} k^{-4} \exp\{-1{\cdot}5\, k_m/k\}$.

spreading factor. For a cross-wind, $\alpha_3 = 90°$, the energy flux is positive for all wavenumbers but quite small, and cannot be drawn on the same scale.

The interaction of a line spectrum with a background continuum can also be investigated using the results of this formalism. Let the line spectrum $F_s(\mathbf{k})$ be approximately represented by a narrow peak at $\mathbf{k}_s$ and let it be superimposed on a smooth 'local sea' spectrum $F(\mathbf{k})$. If the total energy of this 'line spectrum' is $U = \iint F_s(\mathbf{k})\, d\mathbf{k}$, the rate of change of energy U can be obtained by integrating (3.5) over a small region around $\mathbf{k}_s$, where we let $\mathbf{k}_3 = \mathbf{k}_s$, then

$$\frac{dU}{dt} = -U \left\{ \int_{-\infty}^{\infty}\!\!\int a^{(+)} \omega_{\mathbf{k}_s} [\omega_{\mathbf{k}_2} F(\mathbf{k}_1) + \omega_{\mathbf{k}_1} F(\mathbf{k}_2)]\, \delta(\omega_{\mathbf{k}_s} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_1})\, d\mathbf{k}_1 \right.$$
$$\left. + 2 \int_{-\infty}^{\infty}\!\!\int a^{(-)} \omega_{\mathbf{k}_s} [\omega_{\mathbf{k}_2} F(\mathbf{k}_1) - \omega_{\mathbf{k}_1} F(\mathbf{k}_2)]\, \delta(\omega_{\mathbf{k}_s} - \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_1})\, d\mathbf{k}_1 \right\}, \quad (4.5)$$

where $$a^{(+)} = \frac{\pi k_s |D^{+,+}_{\mathbf{k}_1,\mathbf{k}_2}|^2}{2\rho \omega^4_{\mathbf{k}_s} k_1 k_2} \quad \text{and} \quad a^{(-)} = \frac{\pi k_s |D^{+,-}_{\mathbf{k}_1,-\mathbf{k}_2}|^2}{2\rho \omega^4_{\mathbf{k}_s} k_1 k_2}.$$

The first integral in (4.5) applies for sum resonant interactions, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_s$, and the second integral is the contribution from difference resonant interactions, $\mathbf{k}_2 - \mathbf{k}_1 = \mathbf{k}_s$ and $\mathbf{k}_1 - \mathbf{k}_2 = \mathbf{k}_s$.

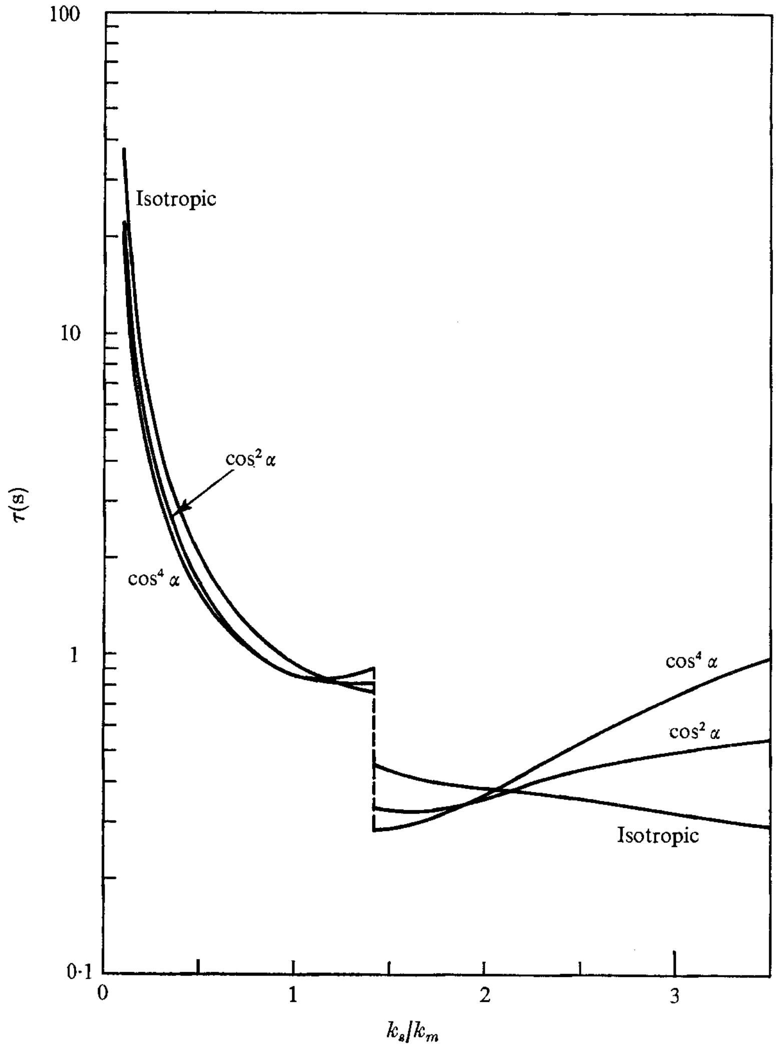

FIGURE 3. Decay time τ for a 'line spectrum' of gravity–capillary waves travelling through a 'local' sea spectrum with different spreading factors. See text.

$$\alpha_s = 0, \quad S(k) = 10^{-2} k^{-4} \exp\{-1{\cdot}5\, k_m/k\}.$$

The solution of (4.5) is

$$U = U_0 \, e^{-t/\tau}, \tag{4.6}$$

where τ^{-1} is given by the integrals in the curly brackets of (4.5).

The interaction time of a line spectrum and a background sea have been obtained for the polar two-dimensional spectrum (4.1); the results are given in figure 3 as a function of spreading factors for $C = 1{\cdot}5$ and in figure 4 the interaction time is plotted as a function of azimuthal direction with respect to the sea direction for a $\cos^2 \alpha$ spreading factor. As expected, the interaction times obtained here are longer than those given by McGoldrick (1965) for discrete interactions. For gravity waves, Hasselmann (1963*b*) also found the interaction times to be

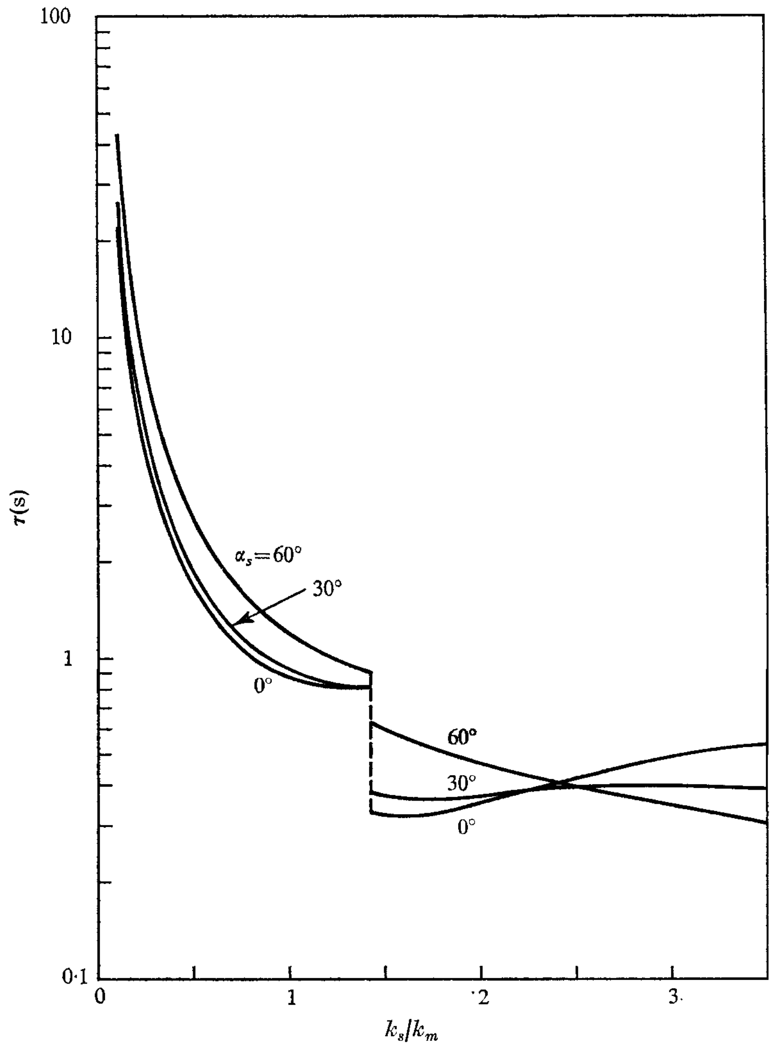

FIGURE 4. Decay time τ for a 'line spectrum' of gravity–capillary waves travelling, in various directions, through a 'local' sea spectrum with a $\cos^2\alpha$ spreading factor.

$$S(\alpha) = 2/\pi \cos^2\alpha, \; S(k) = 10^{-2}k^{-4}\exp\{-1{\cdot}5\, k_m/k\}.$$

greater than those predicted by Phillips (1960) for discrete interactions. The dependence of the energy flux and the decay time of a line spectrum on the stage of development of the sea is illustrated in figures 5 and 6, respectively. As observed, these parameters are quite sensitive to the growth of the sea spectrum; the applicability of the inviscid results will be discussed in the next section.

5. The effect of viscosity

For discrete resonant interactions McGoldrick (1965) verified that, in the first approximation, the effect of viscosity is to introduce a damping factor and a phase change in each wave component participating in the interaction. In

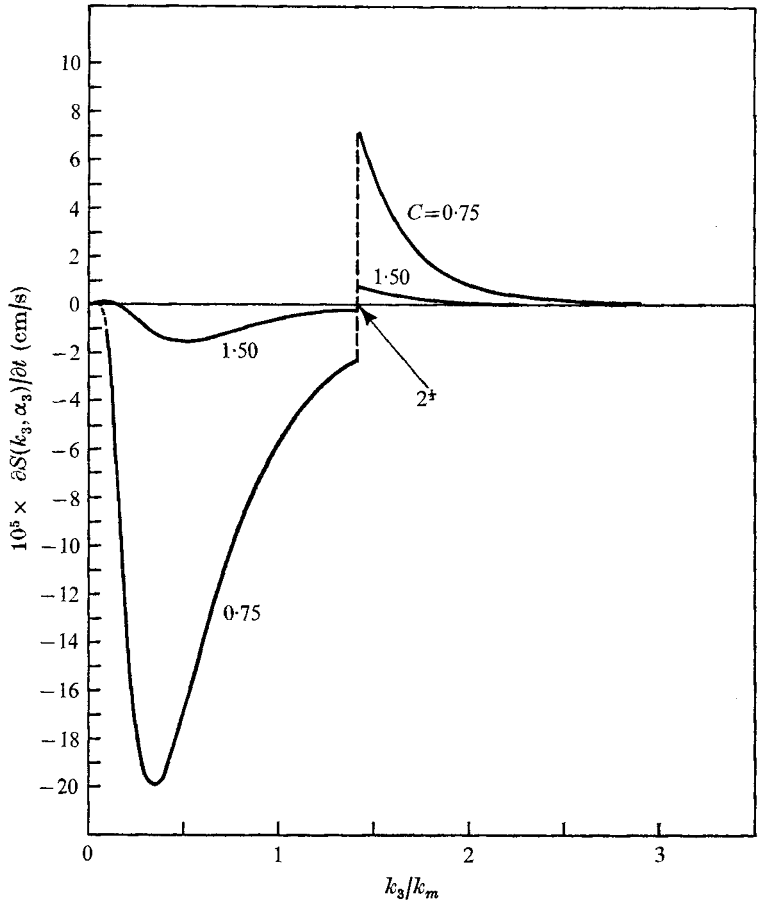

FIGURE 5. Energy transfer for a gravity–capillary wave spectrum in various stages of development. $\alpha_3 = 0$, $S(\alpha) = 2/\pi \cos^2 \alpha$, $S(k) = 10^{-2} k^{-4} \exp\{-Ck_m/k\}$.

addition, McGoldrick found that viscosity may not be significant for wave-numbers between $0 \cdot 7\, k_m$ and about $10\, k_m$ for sum interactions having $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ and $\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} = \omega_{\mathbf{k}_3}$ with $k_1 = k_2$.

However, in the continuous case investigated in the previous sections, the major contribution to the energy transfer and interaction (decay) time arises from wave components which are nearly parallel and of different wavenumbers, in most cases. For example, consider the case of difference interactions in which $\mathbf{k}_2 - \mathbf{k}_1 = \mathbf{k}_3$ and, of course, $\omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_1} = \omega_{\mathbf{k}_3}$ is also satisfied; in this case, when k_3 is in the gravity region both k_1 and k_2 may be in the capillary region and for k_3 in the capillary region k_2 may also be in the capillary region, with k_1 in the gravity region.

Thus in the continuous case the effect of viscosity must be investigated anew. One obvious constraint is that the time constant T_ν of viscous damping of the wave of largest wavenumber participating must be greater than the wave period T_w of the wave of smallest wavenumber. These time constants for the wave amplitudes are given by

$$T_\nu = (2\nu k^2)^{-1} \tag{5.1}$$

and

$$T_w = 2\pi(gk + Tk^3)^{-\frac{1}{2}}, \tag{5.2}$$

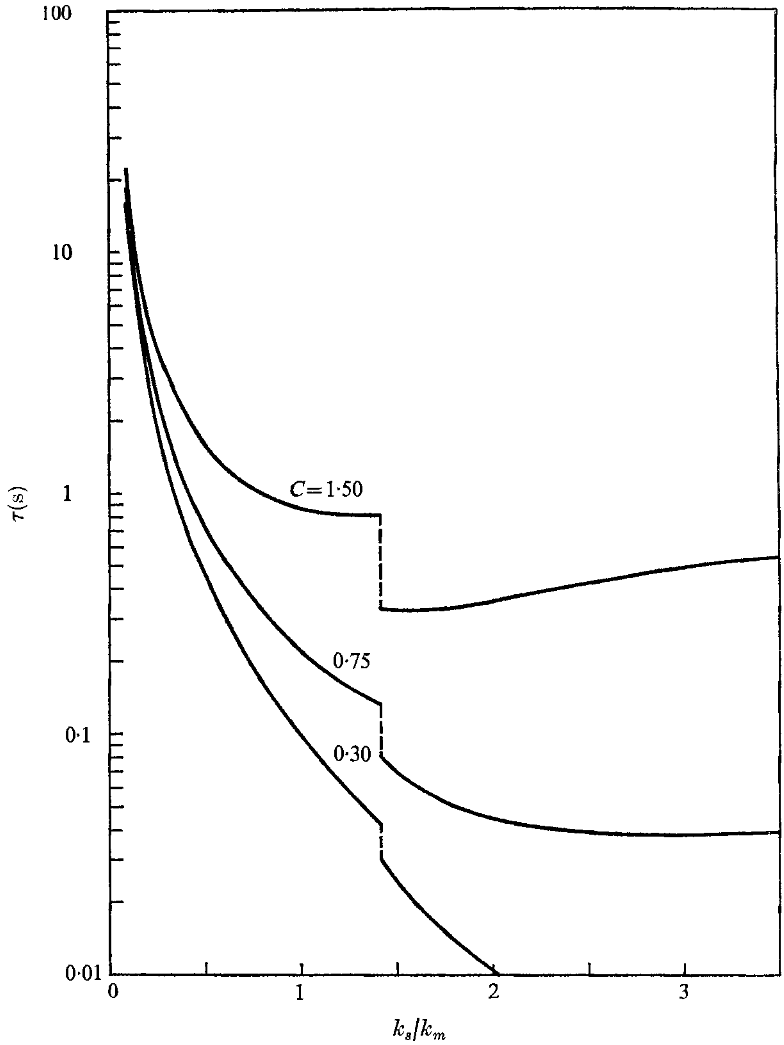

FIGURE 6. Decay time τ for a 'line spectrum' of gravity–capillary waves travelling through a 'local' sea spectrum in various stages of development.

$$\alpha_s = 0, \quad S(\alpha) = 2/\pi \cos^2 \alpha, \quad S(k) = 10^{-2} k^{-4} \exp\{-C k_m/k\}.$$

where ν is the kinematic viscosity. These time constants are shown plotted versus wavenumber in figure 7. A second constraint for the validity of the inviscid results is imposed on the interaction (decay) time 2τ (the factor of two is necessary because this time constant was defined in terms of energy); this should be greater than the wave period of the wave component of the smallest wavenumber participating in the interaction.

The above constraints restrict the validity of the inviscid results for the energy transfer and decay rates to spectra which are in their initial stages of development. For this reason, in the numerical results obtained in the previous section, the maximum energy in the spectra, for the linear approximation for the sea, was selected to occur at wavenumbers of $0{\cdot}375\, k_m$, $0{\cdot}1875\, k_m$ and $0{\cdot}075\, k_m$,

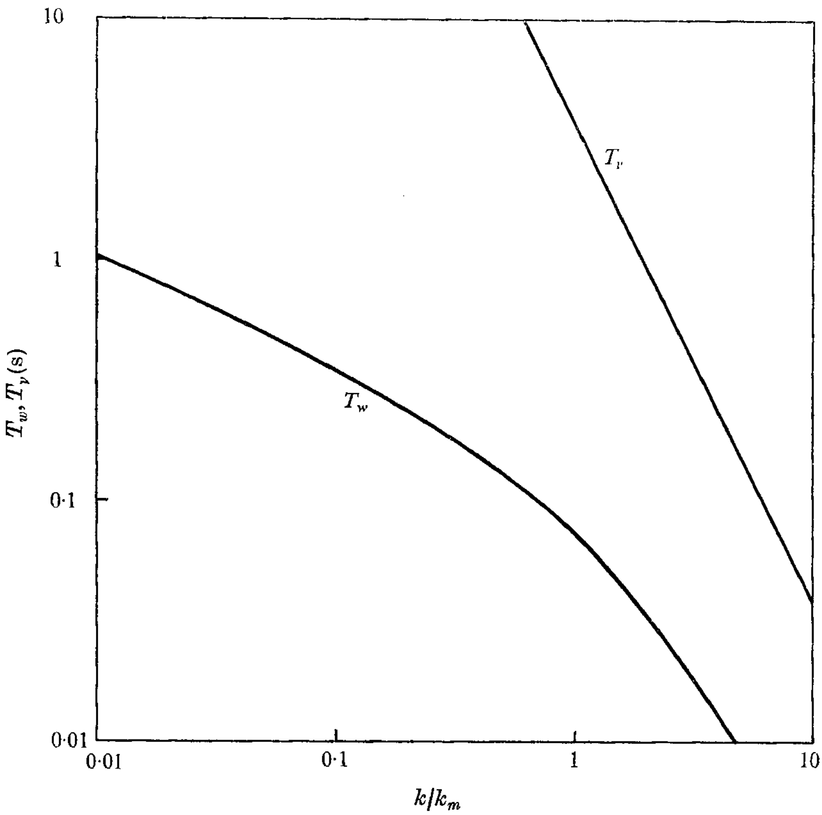

FIGURE 7. Characteristic times of gravity–capillary wave periods (T_w) and viscous damping (T_ν). $\nu = 10^{-2}\,\text{cm}^2/\text{s}$.

respectively. For these spectra the inviscid theory is only applicable for capillary wave components with wavenumbers up to $3\,k_m$ or $4\,k_m$.

The numerical results given in the previous section were obtained for integrations of k_1 from 0 to $20\,k_m$. However, in an investigation of the separate contributions it was found that the energy transfer and decay times were negligible for $k_1 > 4\,k_m$ for $k_3 > 0{\cdot}1\,k_m$. For $k_3 = 0{\cdot}1\,k_m$ the energy transfer vanished for $k_1 < 4\,k_m$, since for this wavenumber the contribution arises from wavenumbers greater than $4\,k_m$.

For cases in which 2τ is smaller than the wave period of the gravity wave at which the maximum energy of the spectrum of the sea occurs, this must be interpreted physically as a broadening of the spectral peak of the line spectrum wave rather than a decay of its energy, as was argued by Hasselmann (1963*b*).

The effect of viscosity on the magnitude of the energy transfer, in first approximation, may be accounted for by introducing the term

$$\partial E/\partial t = -4\nu k_3^2 E. \qquad (5.3)$$

Thus, for large wavenumbers the viscous term should predominate in the energy flux, and viscosity should serve as an energy sink in the overall balancing process. Of course, for more exact results, viscosity must be included *ab initio* in the analysis.

6. Conclusions

We have obtained the energy transfer in gravity–capillary wave spectra using Hasselmann's (1962) inviscid perturbation analysis. In our case the resonant interactions occur at second order because of surface tension. As expected, the energy is transferred from two active wave components to a third passive wave component.

The shape of the energy-transfer curve is similar to that found for gravity waves; energy is transferred from intermediate wavelengths, now in the neighbourhood of 1·7 cm, toward waves of smaller and larger wavelengths. Evident in the energy-transfer curve is a discontinuity for wavenumbers of $2^{\frac{1}{2}} k_m$ which seems to be connected with the instability of 2·44 cm wavelengths. From the results obtained, the 'dip' in the spectrum found for wavelengths in the neighbourhood of 1·7 cm and light winds may be a result of energy transfer because of wave–wave interactions.

The applicability of the inviscid results for the energy flux and interaction time for a line spectrum in a 'background' sea is determined from the condition that both the time constant of viscous decay of the wavenumber participating and the resulting interaction (decay) time be greater than the wave period of the gravity wave of maximum energy in the spectrum of the sea.

The new results should shed additional light in the growth of the spectrum in its early stages. In a more exact analysis viscosity should be introduced in a more formal manner, but the mathematics will be a great deal more complex.

The authors are indebted to Professor Klaus Hasselmann for valuable comments and suggestions during this investigation.

REFERENCES

Benney, D. J. 1962 *J. Fluid Mech.* **14**, 577–584.
Cox, C. X. 1958 *J. Mar. Res.* **16**, 199–225.
Hasselmann, K. 1962 *J. Fluid Mech.* **12**, 481–500.
Hasselmann, K. 1963*a* *J. Fluid Mech.* **15**, 273–281.
Hasselmann, K. 1963*b* *J. Fluid Mech.* **15**, 385–398.
Hasselmann, K. 1966 *Rev. Geophys.* **4**, 1–32.
Hasselmann, K. 1968 *Basic Developments in Fluid Dynamics* (ed. M. Holt), pp. 117–182. Academic.
Longuet-Higgins, M. S. 1962 *J. Fluid Mech.* **12**, 321–332.
McGoldrick, L. F. 1965 *J. Fluid Mech.* **21**, 305–331.
McGoldrick, L. F. 1970 *J. Fluid Mech.* **40**, 251–271.
Phillips, O. M. 1957 *J. Fluid Mech.* **2**, 417–445.
Phillips, O. M. 1960 *J. Fluid Mech.* **9**, 193–217.
Pierson, W. J. & Fife, P. 1961 *J. Geophys. Res.* **66**, 163–179.
Simmons, W. F. 1969 *Proc. Roy. Soc.* A **309**, 551–575.
Valenzuela, G. R., Laing, M. B. & Daley, J. C. 1971 *J. Mar. Res.* **29**, 69–84.
Wright, J. W. & Keller, W. C. 1971 *Phys. Fluids*, **14**, 466–474.